

A UNIQUE CONTINUATION PROPERTY ON THE BOUNDARY FOR SOLUTIONS OF ELLIPTIC EQUATIONS

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ABSTRACT. We prove the following conclusion: if u is a harmonic function on a smooth domain Ω in R^n , $n \geq 3$, or a solution of a general second-order linear elliptic equation on a domain Ω in R^2 , and if there are $x_0 \in \partial\Omega$ and constants $a, b > 0$ such that $|u(x)| \leq a \exp\{-b/|x-x_0|\}$ for $x \in \Omega$, $|x-x_0|$ small, then $u = 0$ in Ω . The decay rate in our results is best possible by the example that $u = \text{real part of } \exp\{-1/z^\alpha\}$, $0 < \alpha < 1$, is harmonic but not identically zero in the right complex half-plane.

1. INTRODUCTION

There are many similar properties between the solutions of elliptic equations and analytic functions. One of the properties is the unique continuation property: if a solution of an elliptic equation (or an analytic function) vanishes up to infinite order at a single interior point of a domain, then the solution (or the analytic function) has to be identically zero. The study of this problem has a long history. For solutions of second-order elliptic equations in the plane, unique continuation was established by Carleman [4]. In 1956–1957, Aronszajn [1] and Cordes [5] extended the unique continuation property to second-order equations in many independent variables. Their work was technical, using Carleman's weighted estimates. Quite recently, an elegant proof of the unique continuation property for solutions of general elliptic equations and systems was found independently by Kazdan [8] and Garofalo-Lin [6].

In this paper, we study a kind of unique continuation property for solutions of an elliptic equation on the boundary: let u be a solution of an elliptic equation $Lu = 0$ in a domain Ω , and x_0 a point on the boundary $\partial\Omega$. If u and all its derivatives $D^\alpha u(x)$ vanish at x_0 (as the limits of $D^\alpha u(x)$ when the point x approaches x_0 , $x \in \Omega$), is u identically zero on Ω ? If we do not assume anything about the rate at which the function u approaches zero at x_0 , then an easy example shows that u might not be zero in the domain: Consider the function $u = \text{real part of } \exp\{-1/z^\alpha\}$, where $0 < \alpha < 1$ and z is a complex variable. It is easy to verify that u is harmonic on the right half-plane, and $D^\alpha u(x)$ approaches zero for all α as the point x approaches 0 from the right half-plane. But in this paper we prove that for harmonic functions and the

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solutions of a general second-order elliptic equation on a 2-dimensional domain, this example illustrates the “ONLY” case that the functions might not be zero. More specifically, we have the following results (the terminology “regular” will be defined in §2):

Theorem 1. *Let Ω be a domain in R^n , $n \geq 2$, and let $x_0 \in \partial\Omega$ with Ω regular at x_0 . If u is a harmonic function on Ω , i.e.,*

$$\Delta u = \sum_{i=1}^{n+1} \frac{\partial^2 u}{\partial x_i^2} = 0 \quad \text{in } \Omega,$$

and, for some constants $a, b > 0$,

$$|u(x)| \leq a \exp \left\{ \frac{-b}{|x - x_0|} \right\} \quad \text{for } x \in \Omega, \quad |x - x_0| \text{ small},$$

then $u = 0$ on Ω .

Theorem 2. *Let Ω be a domain in R^2 , and let $(x_0, y_0) \in \partial\Omega$ with Ω regular at (x_0, y_0) . If $u(x, y)$ is a solution of an elliptic equation*

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu = 0 \quad \text{in } \Omega,$$

where the coefficients are in $C^1(\overline{\Omega})$, and there exist constants $a, b > 0$ such that

$$|u(x, y)| \leq a \exp \left\{ -\frac{b}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right\}$$

for $(x, y) \in \Omega$, (x, y) near (x_0, y_0) , then $u = 0$ in Ω .

The idea of the proof is quite simple. For Theorem 1, using the fact that we are dealing with the standard Laplace operator, we can reduce the problem to the case that the domain Ω is the upper half-space; then again the standard Laplace operator allows us to carry out some calculations which assure us that the harmonic function is zero. For Theorem 2, by the Bers-Nirenberg representation theorem for solutions of a Beltrami-type equation [3], we can reduce the general case to the standard Laplace operator case. Unfortunately, we cannot prove the corresponding version of Theorem 2 in higher dimensional cases. It seems that a different kind of method for the proof of Theorem 1 is needed if we want to prove a similar result for general higher dimensional cases.

Some of the motivation for our research comes from the attempt to understand the unique continuation property for critical points of the p -functional $\int_{\Omega} |\nabla u|^p dx$ (see [8]). The hint is based on the following “observation”: if u is a critical point of the p -functional in a domain Ω , then, if $\nabla u \neq 0$ in Ω_1 for a subdomain Ω_1 in Ω , we have

$$\sum_{i,j=1}^n \left(|\nabla u|^{p-2} \delta_{ij} + (p-2) |\nabla u|^{p-4} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) \frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \quad \text{in } \Omega_1,$$

which is equivalent to a uniformly elliptic equation in Ω_1 . It seems that a suitable version of Theorem 2 for higher dimensional cases could conclude $u = 0$ in Ω_1 ; then we have the unique continuation property for the critical points of the p -functional.

The paper is organized as follows: the proof of Theorem 1 will be given in §2, and in §3 we prove Theorem 2.

2. THE STANDARD LAPLACE OPERATOR CASE

In this section, we will prove Theorem 1.

“ Ω is regular at x_0 ” is defined as follows.

Definition. Let Ω be a domain in R^n , $n \geq 2$, and let x_0 be a point on the boundary of Ω . We say Ω is *regular* at x_0 if there is a ball $B \subset \Omega$ such that $\partial B \cap \partial\Omega = \{x_0\}$.

The proof of the theorem will be divided into several lemmas.

Lemma 2.1. Let $h \in C^0(R^+)$, $R^+ = \{t | t > 0\}$. If, for some constants $a, b > 0$,

$$(2.1) \quad |h(t)| \leq a \exp\{-bt\} \quad \text{for } t > 0$$

and

$$(2.2) \quad \int_0^\infty h(t)t^{2k} dt = 0 \quad \text{for all integer } k \geq 0,$$

then $h(t) = 0$.

Proof. We only have to prove

$$(2.3) \quad \int_0^\infty h(t)e^{-\lambda t} dt = 0 \quad \text{for real } \lambda > 0.$$

In fact, if (2.3) is true, then as the inverse Laplace transformation of 0, $h(t) = 0$ for $t > 0$.

For the proof of (2.3), set

$$F(\alpha) = \int_0^\infty h(t)e^{-\alpha t} dt, \quad \alpha \text{ is a complex variable,}$$

$$g(\beta) = \int_0^\infty h(t) \cos \beta t dt, \quad \beta \text{ is a real variable.}$$

By (2.1), it is easy to see that

$$(2.4) \quad F(\alpha) \text{ is analytic on } \{\alpha | \operatorname{Re} \alpha > -b/2\}$$

and

$$(2.5) \quad g(\beta) \text{ is real analytic on } R^1.$$

For example, (2.4) is an immediate consequence of the following inequality: for $k \geq 0$ and $\operatorname{Re} \alpha > -b/2$,

$$|F^{(k)}(\alpha)| = \left| (-1)^k \int_0^\infty h(t)e^{-\alpha t} t^k dt \right| \leq \int_0^\infty |e^{-\alpha t}| |h(t)| t^k dt$$

$$\leq a \int_0^\infty e^{-bt/2} t^k dt = a 2^{k+1} \frac{k!}{b^{k+1}}.$$

Now (2.2) implies

$$F^{(2k)}(0) = \int_0^\infty h(t)t^{2k} dt = 0 \quad \text{for } k \geq 0.$$

Therefore, for small and real β ,

$$\begin{aligned} g(\beta) &= \operatorname{Re} \left\{ \int_0^\infty h(t) e^{-i\beta t} dt \right\} = \operatorname{Re}\{F(i\beta)\} \\ &= \operatorname{Re} \left\{ \sum_{k=0}^\infty F^{(2k+1)}(0) (-1)^k \frac{\beta^{2k+1}}{(2k+1)!} i \right\} = 0. \end{aligned}$$

Then $g(\beta) = 0$ for all β by (2.5), that is, $\operatorname{Re}\{F(i\beta)\} = 0$ for all β . In addition, we notice that

$$|F(\alpha)| \leq \int_0^\infty |h(t)| dt < \infty \quad \text{for } \operatorname{Re} \alpha \geq 0.$$

Therefore, $w(x, y) = \operatorname{Re} F(x + iy)$ is a bounded harmonic function on $\{(x, y) | x > 0\}$, and $w(0, y) = 0$ for all y . It is clear that $w = 0$ on $\{(x, y) | x > 0\}$. Hence F is a constant function. But $F(0) = 0$ by (2.2), so we have $F(\alpha) = 0$ for $\operatorname{Re} \alpha > 0$. In particular,

$$F(\lambda) = \int_0^\infty h(t) e^{-\lambda t} dt = 0 \quad \text{for all real } \lambda,$$

which is (2.3). Q.E.D.

Lemma 2.2. Let $f \in C^\infty(R^n \setminus \{0\}) \cap C^0(R^n)$,

$$|f(y)| \leq c|y|^{1-n} \quad \text{for } y \in R^n, \quad |y| \text{ large},$$

and

$$u(x, t) = \int_{R^n} \frac{f(y)t}{(|x-y|^2 + t^2)^{(n+1)/2}} dy.$$

If, for some $a, b > 0$,

$$|u(x, t)| \leq a \exp \left\{ \frac{-b}{\sqrt{|x|^2 + t^2}} \right\} \quad \text{for } |x|^2 + t^2 \text{ small}, t > 0,$$

then $u = 0$.

Proof. Since $f \in C^0(R^n)$ and $|f(y)| \leq c|y|^{1-n}$ for $|y|$ large, we see $u(x, 0) = f(x)$ for $x \in R^n$. Then

$$|f(y)| = |u(y, 0)| \leq a \exp \left\{ -\frac{b}{|y|} \right\} \quad \text{for } |y| \text{ small}.$$

We claim

$$(2.6) \quad \int_{R^n} f(y) \frac{\partial}{\partial y^\alpha} (|y|^{-(n+1)}) dy = 0$$

for any n -tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$.

Assuming the claim, the proof of the lemma goes as follows: By (2.6), we have that for any α and integer $k \geq 0$,

$$(2.7) \quad \int_{R^n} f(y) \frac{\partial}{\partial y^\alpha} (\Delta^k (|y|^{-(n+1)})) dy = 0.$$

But an easy computation shows

$$(2.8) \quad \Delta^k (|y|^{-(n+1)}) = c(k, n) |y|^{-(n+1)-2k},$$

where $c(k, n)$ are constants depending only on k and n . Therefore,

$$(2.9) \quad \int_{R^n} f(y)|y|^{-(n+1)-2k} dy = 0 \quad \text{for } k \geq 0.$$

If we set

$$h(r) = \int_{S^{n-1}} f(r\zeta) d\zeta \quad \text{for } r \geq 0,$$

then we have

$$\begin{aligned} \int_0^\infty h(r)r^{-2-2k} dr &= \int_0^\infty \left(\int_{S^{n-1}} f(r\zeta) d\zeta \right) r^{-(n+1)-2k+n-1} dr \\ &= \int_{R^n} f(y)|y|^{-(n+1)-2k} dy = 0 \end{aligned}$$

and

$$|h(r)| \leq \int_{S^{n-1}} |f(r\zeta)| d\zeta \leq a\omega_n \exp\left\{-\frac{b}{r}\right\} \quad \text{for } r \text{ small};$$

$h(r) \rightarrow 0$ as $r \rightarrow \infty$ since $|f(r\zeta)| \leq cr^{1-n}$ for r large. Let $g(r) = h(\frac{1}{r})$; then $|g(r)| \leq a' \exp\{-br\}$ for r large, $g \in C^0([0, \infty))$, and $\int_0^\infty g(r)r^{2k} dr = 0$ for $k \geq 0$.

Now applying Lemma 2.1, we have $g(r) = 0$ for $r \in (0, \infty)$, i.e.,

$$(2.10) \quad \int_{S^n} f(r\zeta) d\zeta = h(r) = 0 \quad \text{for } r \geq 0.$$

On the other hand, (2.6) also implies the following statement: For any tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, there is an integer $k_0(|\alpha|)$ such that, for all $k > k_0$,

$$(2.11) \quad \int_{R^n} f(y)y_1^{\alpha_1}y_2^{\alpha_2}\dots y_n^{\alpha_n}|y|^{-(n+1)-2k} dy = 0.$$

Equality (2.11) can be proved by the following induction:

(i) If $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n = 0$ and $k_0 = 0$, then by (2.9) we have

$$\int_{R^n} f(y)|y|^{-(n+1)-2k} dy = 0 \quad \text{for } k \geq 0.$$

(ii) Assume (2.11) is true for $|\alpha| \leq m$; we want to prove it is true for $|\alpha| = m + 1$. For $|\alpha| = m + 1$, we have the identity

$$(2.12) \quad \begin{aligned} \frac{\partial^\alpha}{\partial y_1^{\alpha_1} \dots \partial y_n^{\alpha_n}} (|y|^{-(n+1)-2k}) &= c(k, n)|y|^{-(n+1)-2k-2|\alpha|} y_1^{\alpha_1} \dots y_n^{\alpha_n} \\ &+ \sum_{|\beta| \leq m} c(\beta, k, n)|y|^{-(n+1)-2k-2\gamma(\beta)} y_1^{\beta_1} \dots y_n^{\beta_n}, \end{aligned}$$

where $c(k, n)$, $c(\beta, k, n)$, and $\gamma(\beta)$ are constants depending on its variables as indicated. This identity also can be proved by induction.

Now let $\gamma_m = \max\{\gamma(\beta) | |\beta| \leq m\}$, and $\bar{k}_m = \max\{k_0(p) | 0 \leq p \leq m\}$. Set $\bar{k} = \gamma_m + \bar{k}_m$, then by the induction assumption,

$$\int_{R^n} f(y)|y|^{-(n+1)-2k-2\gamma(\beta)} y_1^{\beta_1} \dots y_n^{\beta_n} dy = 0$$

for all $k \geq \bar{k}$ and $|\beta| \leq m$. Then (2.7), (2.8), and (2.12) imply (2.11), where $k_0(m+1) = \bar{k}$.

Now by (2.11) and a similar argument to the proof of (2.10), we have: For any monomial $\zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \cdots \zeta_n^{\alpha_n}$,

$$\int_{|\zeta|=1} f(r\zeta) \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \cdots \zeta_n^{\alpha_n} d\zeta = 0 \quad \text{for } r > 0.$$

Therefore, $f(r\zeta) = 0$ for all $r > 0$ and $\zeta \in S^{n-1}$, thus $u = 0$.

In order to complete the proof of Lemma 2.2, we have to verify our claim (2.6). In fact, if we pretend that $u \in C^\infty(\overline{R_+^{n+1}})$, and if we take derivatives under the integral and evaluate them at $t = 0$, we immediately get (2.6). But in our case, we only assume $f \in C^\infty(R^n \setminus \{0\})$, and u may not be in $C^\infty(\overline{R_+^{n+1}})$. Thus we have to verify (2.6) carefully. The idea is to use the growth assumption on u to analyze the limits $D^\alpha u(x, t)$ as $|x|^2 + |t|^2$ tends to zero. The detailed proof goes as follows:

First, we claim that if $x_0 \in R^n$, $t_0 > 0$, and $|x_0| < 1$, then

- (i) $\lim_{s \rightarrow 0^+} \frac{\partial^\alpha u(sx_0, st_0)}{\partial x^\alpha} = 0$ for all $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$,
- (ii) $\lim_{s \rightarrow 0^+} \frac{\partial^{\alpha+1} u(sx_0, st_0)}{\partial x^\alpha \partial t} = (-1)^{|\alpha|} \int_{R^n} f(y) \frac{\partial}{\partial y^\alpha} (|y|^{-(n+1)}) dy.$

In fact,

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} &= \int_{R^n} f(y) t \frac{\partial}{\partial x^\alpha} \left[\frac{1}{(|x - y|^2 + t^2)^{(n+1)/2}} \right] dy \\ &= (-1)^{|\alpha|} \int_{R^n} f(y) t \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|x - y|^2 + t^2)^{(n+1)/2}} \right] dy. \end{aligned}$$

Then

$$\frac{\partial^\alpha u(sx_0, st_0)}{\partial x^\alpha} = (-1)^{|\alpha|} st_0 \int_{R^n} f(y) \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+1)/2}} \right] dy.$$

Now (i) is clear if we can prove that for some constant $c(\alpha, x_0, t_0)$

$$(2.13) \quad \left| \int_{R^n} f(y) \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+1)/2}} \right] dy \right| < c(\alpha, x_0, t_0).$$

Inequality (2.13) follows from

$$(2.14) \quad \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+1)/2}} \right] = \sum_{\text{finite sum}} \frac{P_k(y - sx_0)}{(|y - sx_0|^2 + s^2 t_0^2)^k}$$

and for $k \geq (n+1)/2$

$$(2.15) \quad \int_{R^n} |f(y)| \frac{1}{(|y - sx_0|^2 + s^2 t_0^2)^k} dy \leq c(x_0, t_0, k),$$

where $P_k(y - sx_0)$ is a polynomial of $y - sx_0$ with coefficients possibly depending on s , x_0 , and t_0 , and $\deg P_k(y - sx_0) = \text{degree of } P_k \text{ as a polynomial of } y$, $\deg P_k(y - sx_0) + n + 1 \leq 2k$.

Equality (2.14) is clear by induction on $|\alpha|$. For (2.15), we have

$$\begin{aligned} \int_{R^n} |f(y)| \frac{1}{(|y - sx_0|^2 + s^2 t_0^2)^k} dy &= \int_{|y| \leq As} |f(y)| \frac{1}{(|y - sx_0|^2 + s^2 t_0^2)^k} dy \\ &\quad + \int_{|y| \geq As} |f(y)| \frac{1}{(|y - sx_0|^2 + s^2 t_0^2)^k} dy \\ &= I_1 + I_2, \quad \text{where } A = 4| |x_0|^2 - t_0^2 | + 1. \end{aligned}$$

But

$$\begin{aligned} I_1 &\leq \left(\frac{1}{s^2 t_0^2} \right)^k \int_{|y| \leq As} |f(y)| dy \leq a \left(\frac{1}{s^2 t_0^2} \right)^k \int_{|y| \leq As} \exp \left\{ -\frac{b}{|y|} \right\} dy \\ &\leq c(a, t_0, k, n, x_0) s^{n-k} \exp \left\{ -\frac{b}{As} \right\} \leq c(x_0, t_0, k) \end{aligned}$$

and

$$\begin{aligned} I_2 &\leq \int_{|y| \geq As} |f(y)| \frac{1}{(|y|^2 + s^2 |x_0|^2 - 2|y||x_0|s + s^2 t_0^2)^k} dy \\ &\leq \int_{|y| \geq As} |f(y)| \frac{4^k}{|y|^{2k}} dy \leq c(k). \end{aligned}$$

Therefore, (2.15) is true.

Now we prove (ii): An easy computation shows

$$\begin{aligned} \frac{\partial^{\alpha+1} u(sx_0, st_0)}{\partial x^\alpha \partial t} &= (-1)^{|\alpha|} \int_{R^n} f(y) \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+1)/2}} \right] dy \\ &\quad + (-1)^{|\alpha|+1} (n+1) s^2 t_0^2 \int_{R^n} f(y) \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+3)/2}} \right] dy. \end{aligned}$$

Then (ii) follows by the statements

$$(2.16) \quad \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+1)/2}} \right] = \sum_{\text{finite sum}} \frac{\tilde{P}_k(y - sx_0)}{(|y - sx_0|^2 + s^2 t_0^2)^k},$$

$$(2.17) \quad \int_R |f(y)| \frac{1}{(|y - sx_0|^2 + s^2 t_0^2)^k} dy \leq c(x_0, t_0, k) \quad \text{for } k \geq \frac{n+3}{2},$$

and

$$\begin{aligned} (2.18) \quad &\lim_{s \rightarrow 0^+} \int_{R^n} f(y) \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+1)/2}} \right] dy \\ &= \int_{R^n} f(y) \frac{\partial}{\partial y^\alpha} (|y|^{-(n+1)}) dy, \end{aligned}$$

where $\tilde{P}_k(y - sx_0)$ is a polynomial of $y - sx_0$ and $\deg \tilde{P}_k + n + 3 \leq 2k$. The proofs of (2.16) and (2.17) are exactly the same as the proofs of (2.14) and

(2.15). Thus we only have to prove (2.18).

$$\begin{aligned}
& \left| \int_{R^n} f(y) \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+1)/2}} \right] dy - \int_{R^n} f(y) \frac{\partial}{\partial y^\alpha} (|y|^{-(n+1)}) dy \right| \\
& \leq \int_{|y| \geq M} |f(y)| \left| \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+1)/2}} \right] \right| dy \\
& \quad + \int_{|y| \leq \delta} |f(y)| \left| \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+1)/2}} \right] \right| dy \\
& \quad + \int_{|y| \geq M} |f(y)| \left| \frac{\partial}{\partial y^\alpha} (|y|^{-(n+1)}) \right| dy \\
& \quad + \int_{|y| \leq \delta} |f(y)| \left| \frac{\partial}{\partial y^\alpha} (|y|^{-(n+1)}) \right| dy \\
& \quad + \int_{\delta \leq |y| \leq M} \left| f(y) \left(\frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+1)/2}} \right] - \frac{\partial}{\partial y^\alpha} (|y|^{-(n+1)}) \right) \right| dy \\
& = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5.
\end{aligned}$$

For $A = 4|x_0|^2 - t_0^2 + 1$, choose $M > A$ large such that $\tilde{I}_3 < \varepsilon/8$ and

$$\begin{aligned}
\tilde{I}_1 & \leq \sum_{\substack{\text{finite} \\ \text{sum} \\ 2k \geq n+1}} c(k, x_0, t_0) \int_{|y| \geq M} |f(y)| \frac{1}{(|y - sx_0|^2 + s^2 t_0^2)^k} dy \\
& \leq \sum c(k, x_0, t_0) \int_{|y| \geq M} |f(y)| \frac{4^k}{|y|^{2k}} dy \leq \frac{\varepsilon}{8}.
\end{aligned}$$

Also take $\delta > 0$ small such that $\tilde{I}_4 < \varepsilon/8$, and for $0 < s < \delta/A$

$$\begin{aligned}
& \int_{As \leq |y| \leq \delta} |f(y)| \left| \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+1)/2}} \right] \right| dy \\
& \leq \sum_{\substack{\text{finite} \\ \text{sum} \\ 2k \geq n+1}} c(k, x_0, t_0) \int_{As \leq |y| \leq \delta} |f(y)| \frac{1}{(|y - sx_0|^2 + s^2 t_0^2)^k} dy \\
& \leq \sum c(k, x_0, t_0) \int_{As \leq |y| \leq \delta} |f(y)| \frac{4^k}{|y|^{2k}} dy \leq \frac{\varepsilon}{8}.
\end{aligned}$$

Now fix δ and M , and choose s_1 such that $s_1 < \varepsilon/A$ and $\tilde{I}_5 < \varepsilon/8$ for $0 < s < s_1$. Take $s_2 < s_1$ so that

$$\begin{aligned}
& \int_{|y| \leq As} |f(y)| \left| \frac{\partial}{\partial y^\alpha} \left[\frac{1}{(|sx_0 - y|^2 + s^2 t_0^2)^{(n+1)/2}} \right] \right| dy \\
& \leq \sum_{\substack{\text{finite} \\ \text{sum} \\ 2k \geq n+1}} c(k, x_0, t_0) \int_{|y| \leq As} |f(y)| \frac{1}{(|y - sx_0|^2 + s^2 t_0^2)^k} dy \\
& \leq a \sum c(k, x_0, t_0) \left(\frac{1}{s^2 t_0^2} \right)^k \exp \left\{ -\frac{b}{As} \right\} (As)^n \omega_n \\
& \leq \frac{\varepsilon}{8} \quad \text{for } 0 < s < s_2.
\end{aligned}$$

Then $\tilde{I}_2 < \varepsilon/8$ for $0 < s < s_2$. This completes the proof of (2.18).

Next we notice that by (i), (ii), and

$$-\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2},$$

for any $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, integer k , and fixed (x_0, t_0) , $t_0 > 0$,

$$\lim_{s \rightarrow 0^+} \frac{\partial^{\alpha+k} u(sx_0, st_0)}{\partial x^\alpha \partial t^k} = U_{\alpha k}$$

exists.

Now for fixed (x_0, t_0) , $t_0 > 0$, $|x_0| < 1$, we have (denote $x_{n+1} = t$ and $x_{n+1}^0 = t_0$)

$$\begin{aligned} u(x_0, t_0) &= u(sx_0, st_0) + (1-s) \sum_{i=1}^{n+1} \frac{\partial u(sx_0, st_0)}{\partial x_i} x_i^0 \\ &\quad + \cdots + \frac{1}{p!} (1-s)^p \sum_{|\alpha|=p} C_\alpha \frac{\partial^\alpha u(sx_0, st_0)}{\partial x^\alpha} x_0^\alpha \\ &\quad + \frac{1}{p!} \int_s^1 \left(\sum_{|\alpha|=p+1} C_\alpha \frac{\partial^\alpha u(zx_0, zt_0)}{\partial x^\alpha} x_0^\alpha \right) (1-z)^p dz, \end{aligned}$$

where $0 < s < 1$ and the C_α are suitable constants. Letting $s \rightarrow 0$, we get

$$\begin{aligned} u(x_0, t_0) &= u_{00} + \sum_{|\alpha|+k=1} u_{\alpha k} x_0^\alpha t_0^k + \frac{1}{2!} \sum_{|\alpha|+k=2} C_{\alpha k} u_{\alpha k} x_0^\alpha t_0^k \\ &\quad + \cdots + \frac{1}{p!} \sum_{|\alpha|+k=p} C_{\alpha k} u_{\alpha k} x_0^\alpha t_0^k \\ &\quad + \frac{1}{p!} \int_0^1 \left(\sum_{|\alpha|+k=p+1} C_{\alpha k} \frac{\partial^{\alpha+k} u(zx_0, zt_0)}{\partial x^\alpha \partial t^k} x_0^\alpha t_0^k \right) (1-z)^p dz. \end{aligned}$$

Since all $u_{\alpha k}$ are independent of x_0 and t_0 , we have

$$\begin{aligned} u(\beta x_0, \beta t_0) &= u_{00} + \beta \sum_{|\alpha|+k=1} u_{\alpha k} x_0^\alpha t_0^k + \frac{1}{2!} \beta^2 \sum_{|\alpha|+k=2} C_{\alpha k} u_{\alpha k} x_0^\alpha t_0^k \\ &\quad + \cdots + \frac{1}{p!} \beta^p \sum_{|\alpha|+k=p} C_{\alpha k} u_{\alpha k} x_0^\alpha t_0^k \\ &\quad + \frac{1}{p!} \beta^{p+1} \int_0^1 \left(\sum_{|\alpha|+k=p+1} C_{\alpha k} \frac{\partial^{\alpha+k} u(\beta zx_0, \beta zt_0)}{\partial x^\alpha \partial t^k} x_0^\alpha t_0^k \right) (1-z)^p dz. \end{aligned}$$

Notice that, for fixed (x_0, t_0) ,

$$\frac{\partial^{\alpha+k} u(\beta zx_0, \beta zt_0)}{\partial x^\alpha \partial t^k} \text{ is bounded for } 0 < \beta < 1, \ 0 < z < 1,$$

and

$$|u(\beta x_0, \beta t_0)| \leq a \exp \left\{ -\frac{b}{\beta} \frac{1}{\sqrt{|x_0|^2 + t_0^2}} \right\}.$$

Then we have

$$\sum_{|\alpha|+k=j} C_{\alpha k} u_{\alpha k} x_0^\alpha t_0^k = 0 \quad \text{for } j = 1, 2, \dots,$$

but (x_0, t_0) can be chosen arbitrary as long as $|x_0| < 1$, and $t_0 > 0$. Hence,

$$u_{\alpha k} = 0 \quad \text{for all } \alpha, k.$$

In particular,

$$0 = u_{\alpha 1} = (-1)^{|\alpha|} \int_{R^n} f(y) \frac{\partial}{\partial y^\alpha} (|y|^{-(n+1)}) dy,$$

which is (2.6). Q.E.D.

Lemma 2.3. Let $B_1(0)$ be a unit ball in R^{n+1} , $n \geq 2$, centered at 0, and let $x_0 = (0, 0, \dots, -1)$. If $u(x_1, \dots, x_{n+1})$ satisfies

$$(1) \Delta u = \sum \partial^2 u / \partial x_i^2 = 0 \text{ in } B, \text{ and}$$

$$(2) |u(x)| \leq a \exp\{-b/|x - x_0|\}$$

for some $a, b > 0$, $x \in B$, and $|x - x_0|$ small, then $u = 0$ in B .

Proof. Set

$$v(x) = |x|^{1-n} u \left(\frac{x}{|x|^2} \right);$$

it is well known that $\Delta v(x) = 0$ in $R^{n+1} \setminus B_1(0)$. Set

$$f(y) = v(y, -1) = (1 + |y|^2)^{(1-n)/2} u \left(\frac{y}{|y|^2 + 1}, \frac{-1}{|y|^2 + 1} \right).$$

It is not difficult to verify:

(a)

$$v(x, -t-1) = c_n \int_{R^n} \frac{f(y)t}{(|x-y|^2 + t^2)^{(n+1)/2}} dy \quad \text{for } t > 0, x \in R^n.$$

(b) For some constants $a_1, b_1 > 0$, if $t > 0$ and $|y|^2 + t^2$ small, then

$$|v(x, -t-1)| \leq a_1 \exp \left\{ -\frac{b_1}{\sqrt{|y|^2 + t^2}} \right\}.$$

(c) $f(y) \in C^\infty(R^n \setminus \{0\}) \cap C^0(R^n)$.

(d) $|f(y)| \leq C|y|^{1-n}$ for $|y| \geq 1$.

Now an application of Lemma 2.2 concludes that $v(x, -t-1) = 0$ for $t > 0$ and $x \in R^n$. Therefore, $u = 0$ on some open set in B , and hence $u = 0$ in B by the fact that u is analytic in B . Q.E.D.

Proof of Theorem 1. Since Ω is regular at x_0 , there is a ball $B \subset \Omega$ such that $\partial B \cap \partial \Omega = x_0$. (This ball is a disc in the case $n = 2$.) Then $\Delta u = 0$ in B , and, for some $a, b > 0$,

$$|u(x)| \leq a \exp \left\{ -\frac{b}{|x - x_0|} \right\} \quad \text{for } x \in B \text{ and } |x - x_0| \text{ small.}$$

It is clear that we may assume $B = B_1(0)$ —a ball centered at the origin with radius 1—and also that $x_0 = (0, \dots, -1)$. Hence, for $n \geq 3$, Lemma 2.3

implies $u = 0$ in B , therefore $u = 0$ in Ω . For $n = 2$, we define a harmonic function u^* on $B \times R$ as

$$u^*(x_1, x_2, x_3) = u(x_1, x_2) \quad \text{for } (x_1, x_2) \in B \text{ and } x_3 \in R.$$

Then it is easy to see that $\Delta u^* = 0$ in $B \times R$ and, for some $a, b > 0$,

$$|u^*(x)| \leq a \exp \left\{ \frac{-b}{|x - x_0^*|} \right\} \quad \text{for } x \in B \times R \text{ and } |x - x_0^*| \text{ small,}$$

where $x_0^* = (x_0, 0)$. Now an application of the result in $n = 3$ yields $u^* = 0$ in $B \times R$, i.e., $u = 0$ in B and $u = 0$ in Ω . Q.E.D.

Now we give an example to show that the decay rate in our result is the best possible.

Example. For $n \geq 2$ and $0 < \alpha < 1$, there is a harmonic function u defined on $R_+^n = \{(x_1, \dots, x_n) | x_n > 0\}$ such that $\Delta u = 0$ on R_+^n and

$$|u(x)| \leq a \exp \left\{ -\frac{b}{|x|^\alpha} \right\} \quad \text{for some } a, b > 0, \quad x \in R_+^n, \text{ and } |x| \text{ small.}$$

Furthermore, u is not identically zero on R_+^n .

Proof. Set

$$u^*(x, y) = \operatorname{Re} \exp \left\{ -\frac{1}{z^\alpha} \right\}, \quad z = x + iy,$$

where z^α is an analytic function on the right complex half-plane.

For $n \geq 2$, set

$$u(x_1, \dots, x_n) = u^*(x_n, x_1) \quad \text{for } (x_1, \dots, x_n) \in R_+^n.$$

It is easy to verify that the $u(x_1, \dots, x_n)$ is what we want. Q.E.D.

Remark. It is not difficult to see that for any quite smooth domain and any point on the boundary of the domain, we can construct a nonzero harmonic function on the domain which is approaching zero at that point in the rates as indicated in the example.

3. THE GENERAL CASE FOR DIMENSION TWO

After we considered the standard Laplace operator, a natural question arose: Can we prove a similar result for the solutions of general elliptic equations? At the present stage, we could only give a satisfactory answer when the dimension of the domain is two. Namely, we can prove Theorem 2, restated here for convenience.

Theorem 2. Let Ω be a domain in R^2 , and let $(x_0, y_0) \in \partial\Omega$ with Ω regular at (x_0, y_0) . If $u(x, y)$ is a solution of an elliptic equation

$$(3.1) \quad a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu = 0 \quad \text{in } \Omega,$$

where the coefficients are in $C^1(\overline{\Omega})$, and there exist constants $a, b > 0$, such that

$$|u(x, y)| \leq a \exp \left\{ -\frac{b}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right\}$$

for $(x, y) \in \Omega$, (x, y) near (x_0, y_0) , then $u = 0$ in Ω .

The proof of the theorem will be divided into several lemmas. Because our assumptions allow us to extend the coefficients of the equation into a neighborhood of x_0 , the proofs of Lemmas 3.1–3.3 are similar to the proofs of the corresponding conclusions for the case of the unique continuation property at an interior point in a domain of dimension two (one may see [3]). Thus we omit the proofs of the lemmas here.

Lemma 3.1. *If a_{ij} , b_i , and c , $1 \leq i \leq 2$, $1 \leq j \leq 2$, are continuous on a closed domain B , and x_0 is a point on ∂B , then there is a neighborhood V of x_0 and a positive function u^* such that*

$$(3.2) \quad a_{11}u_{xx}^* + 2a_{12}u_{xy}^* + a_{22}u_{yy}^* + b_1u_x^* + b_2u_y^* + cu^* = 0 \quad \text{in } B \cap V.$$

Lemma 3.2. *If u is a solution of (3.1) in B which is a closed domain, and a positive function u^* satisfies (3.2) in $V \cap B$, then the function $v = u/u^*$ satisfies the following equation:*

$$\begin{aligned} & a_{11}v_{xx} + 2a_{12}v_{xy} + a_{22}v_{yy} \\ & + \left(b_1 + \frac{1}{u^*}(2a_{11}u_x^* + 2a_{12}u_y^*) \right) v_x \\ & + \left(b_2 + \frac{1}{u^*}(2a_{12}u_x^* + 2a_{22}u_y^*) \right) v_y = 0 \quad \text{in } B \cap V. \end{aligned}$$

Lemma 3.3. *If u is a solution of (3.1) in B which is a closed domain, $(0, 0) \in \partial B$, and all coefficients a_{ij} , $1 \leq i, j \leq 2$, are in $C^1(B \cap V_0)$, where V_0 is a neighborhood of $(0, 0)$, then there is a diffeomorphism $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ which is defined in a neighborhood V_1 of $(0, 0)$ such that $v(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$ satisfies*

$$(3.3) \quad v_{\xi\xi} + v_{\eta\eta} + \tilde{b}_1v_\xi + \tilde{b}_2v_\eta + \tilde{c}v = 0 \quad \text{in } \tilde{V}_1,$$

where \tilde{V}_1 is the image of V_1 under the diffeomorphism $(x, y) \rightarrow (\xi, \eta)$. Furthermore, if $c = 0$ in (3.1), then $\tilde{c} = 0$ in (3.3).

Lemma 3.4. *If B is a domain with smooth boundary ∂B , $(0, 0) \in \partial B$, and u is a function satisfying*

$$\begin{aligned} & \Delta u + b_1u_x + b_2u_y = 0 \quad \text{in } B, \\ & |u(x, y)| \leq a \exp \left\{ -\frac{b}{\sqrt{(x^2 + y^2)}} \right\} \quad \text{for some } a, b > 0, (x, y) \in B, \end{aligned}$$

then there is a ball $B^* \subset B$ and constants $a', b' > 0$ such that $B \cap B^* = (0, 0)$ and

$$\begin{aligned} & |u_x(x, y)| \leq a' \exp \left\{ -\frac{b'}{\sqrt{(x^2 + y^2)}} \right\} \quad \text{in } B^*, \\ & |u_y(x, y)| \leq a' \exp \left\{ -\frac{b'}{\sqrt{(x^2 + y^2)}} \right\} \quad \text{in } B^*. \end{aligned}$$

Proof. We may assume that B is a closed ball. It is easy to see that there is a diffeomorphism $\Phi: B \rightarrow D$, where D is a closed domain in R^2 , and

$D \supset B_+((0, 0), 4)$; here $B_+((0, 0), 4)$ is a closed half-ball in the upper half-space. Under the coordinate $(\xi, \eta) = \Phi(x, y)$, our equation looks like

$$a_{11}(\xi, \eta)u_{\xi\xi} + 2a_{12}(\xi, \eta)u_{\xi\eta} + a_{22}(\xi, \eta)u_{\eta\eta} + b_1(\xi, \eta)u_\xi + b_2(\xi, \eta)u_\eta = 0,$$

and this equation is uniformly elliptic on $B_+((0, 0), 4)$, i.e., there are λ and Λ such that

$$(a_{ij}) \geq \lambda I \quad \text{on } B_+((0, 0), 4), \\ |a_{ij}|_{0, \alpha} \leq \Lambda, \quad |b|_{0, \alpha} \leq \Lambda.$$

By the interior estimates for solutions of an elliptic equation (see [7]), there is a constant $C = C(\lambda, \Lambda, \alpha)$ such that, for $0 < t \leq 1$ and $|s| \leq 1$,

$$\frac{t}{4} \sup \left\{ |Du(\xi, \eta)| \mid (\xi, \eta) \in B \left((s, t), \frac{t}{4} \right) \right\} \\ \leq C \sup \left\{ |u(\xi, \eta)| \mid (\xi, \eta) \in B \left((s, t), \frac{t}{2} \right) \right\}.$$

In particular,

$$|Du(s, t)| \leq \frac{4C}{t} \sup \left\{ |u(\xi, \eta)| \mid (\xi, \eta) \in B \left((s, t), \frac{t}{2} \right) \right\}.$$

Now if $(\xi - s)^2 + (\eta - t)^2 \leq t^2/4$, it is easy to check that

$$-\frac{1}{\sqrt{(\xi^2 + \eta^2)}} \leq -\frac{1}{\sqrt{3(s^2 + t^2)}}.$$

Therefore,

$$\sup \left\{ |u(\xi, \eta)| \mid (\xi, \eta) \in B \left((s, t), \frac{t}{2} \right) \right\} \leq a \exp \left\{ -\frac{b}{\sqrt{3}} \frac{1}{\sqrt{s^2 + t^2}} \right\},$$

hence

$$|Du(s, t)| \leq \frac{4aC}{t} \exp \left\{ -\frac{b}{\sqrt{3}} \frac{1}{\sqrt{s^2 + t^2}} \right\}.$$

But if $s^2 + (t - 1)^2 \leq 1$, $0 < t < 1$, then we have $t \geq s^2$ and $2t \geq t^2 + s^2$, thus

$$|Du(s, t)| \leq \frac{4aC}{t} \exp \left\{ -\frac{b}{\sqrt{3}} \frac{1}{\sqrt{s^2 + t^2}} \right\} \\ \leq \frac{8aC}{(s^2 + t^2)} \exp \left\{ -\frac{b}{\sqrt{3}} \frac{1}{\sqrt{s^2 + t^2}} \right\} \\ \leq C_0 \exp \left\{ -\frac{b}{2\sqrt{3}} \frac{1}{\sqrt{s^2 + t^2}} \right\}$$

for some $C_0 = C_0(\lambda, \Lambda, \alpha, a, b)$. Finally, using the diffeomorphism, we easily see that the conclusion of the lemma is true. Q.E.D.

Proof of Theorem 2. We only have to prove that $u = 0$ in some open set in Ω .

First, we may assume $(x_0, y_0) = (0, -1)$ and $\Omega = B$ is a ball. If necessary, we may replace B by some smaller balls which attach to B from the inside at $(0, -1)$, and by Lemmas 3.1–3.4 we may assume

$$\begin{aligned}\Delta z + b_1 z_x + b_2 z_y &= 0 \quad \text{in } B, \\ |z(x, y)| &\leq a \exp \left\{ -\frac{b}{\sqrt{x^2 + (y+1)^2}} \right\} \quad \text{in } B, \\ |Dz(x, y)| &\leq a \exp \left\{ -\frac{b}{\sqrt{x^2 + (y+1)^2}} \right\} \quad \text{in } B,\end{aligned}$$

where a and b are constants and z is a function related to u in this way such that $z = 0$ in B is equivalent to $u = 0$ in B .

Set $w = z_x$ and $v = z_y$; then

$$(3.4) \quad w_x = v_y - b_1 w + b_2 v, \quad -w_y = v_x.$$

Also, set

$$\begin{aligned}q = w + iv &= z_x - iz_y, \quad \bar{q} = \frac{1}{2} \left(\frac{\partial q}{\partial x} + i \frac{\partial q}{\partial y} \right), \\ \bar{q} &= w - iv, \quad \alpha = -\frac{b_1}{4} - i \frac{b_2}{4}, \quad \beta = -\frac{b_1}{4} + i \frac{b_2}{4}.\end{aligned}$$

It is easy to verify that (3.4) is equivalent to the equation

$$(3.5) \quad q_{\bar{z}} = \alpha q + \beta \bar{q}.$$

Now by a representation result due to Bers-Nirenberg (see [3]), we have

$$q(\lambda) = e^{s(\lambda)} f(\lambda) \quad \text{for } \lambda = \xi + i\eta \in B_1(0) \text{ in the complex plane,}$$

where $s(\lambda)$ is continuous on $\overline{B_1(0)}$ and real on $|\lambda| = 1$, and $f(\lambda)$ is an analytic function on $B_1(0)$.

Since $s(\lambda)$ is continuous on $\overline{B_1(0)}$, there is a constant $c_1 > 0$ such that $|e^{-s(\lambda)}| \leq c_1$ for $\lambda \in B_1(0)$. Hence

$$|f(\lambda)| \leq c_1 |q| \leq c_1 \sqrt{z_x^2 + z_y^2}.$$

By the conditions satisfied by z , we see that

$$|f(\lambda)| \leq c_2 \exp \left\{ -\frac{b}{|\lambda + i|} \right\}.$$

Then our result for the standard Laplace case implies that $f(\lambda)$ is identically zero on $B_1(0)$, i.e., $z_x = z_y = 0$ on $B_1(0)$. But $z(0, -1) = 0$, therefore z vanishes on $B_1(0)$. As we remarked, u is identically zero on B and Ω , which completes the proof of our result. Q.E.D.

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